ISSN 2348-1196 (print)
International Journal of Computer Science and Information Technology Research ISSN 2348-120X (online)
Vol. 11, Issue 1, pp: (22-27), Month: January - March 2023, Available at: www.researchpublish.com

Evaluating an Improper Fractional Integral Based on Jumarie's Modified Riemann-Liouville Fractional Calculus

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DOI: https://doi.org/10.5281/zenodo.7551811

Published Date: 19-January-2023

Abstract: **In this paper, based on Jumarie's modified Riemann-Liouville (R-L) fractional calculus, we find the solution of some type of improper fractional integral. Change of variable for fractional calculus, integration by parts for fractional calculus, fractional L'Hospital's rule, and a new multiplication of fractional analytic functions play important roles in this article. In fact, our result is a generalization of the traditional calculus result.**

Keywords: **Jumarie's modified R-L fractional calculus, improper fractional integral, change of variable, integration by parts, fractional L'Hospital's rule, new multiplication, fractional analytic functions.**

I. INTRODUCTION

Fractional calculus with derivatives and integrals of any real or complex order has its origin in the work of Euler, and even earlier in the work of Leibniz. Shortly after being introduced, the new theory turned out to be very attractive to many famous mathematicians and scientists, for example, Laplace, Riemann, Liouville, Abel, and Fourier. Fractional calculus has important applications in many scientific fields such as physics, mechanics, biology, engineering, viscoelasticity, dynamics, control theory, economics, and so on [1-11].

However, the definition of fractional derivative is not unique. Commonly used definitions include Riemann-Liouville (R-L) fractional derivative, Caputo fractional derivative, Grunwald-Letnikov (G-L) fractional derivative, Jumarie's modified R-L fractional derivative [12-16]. Since Jumarie type of R-L fractional derivative helps to avoid non-zero fractional derivative of constant function, it is easier to use this definition to connect fractional calculus with classical calculus.

In this paper, based on Jumarie type of R-L fractional calculus, we find the solution of the following improper α-fractional integral:

$$\left({}_{0}I^{\alpha}_{[\Gamma(\alpha+1)]^{\frac{1}{\alpha}}}\right)\left[\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes-\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)}\right], \quad (1)$$

where $0<\alpha\leq 1$. Change of variable for fractional calculus, integration by parts for fractional calculus, fractional L'Hospital's rule, and a new multiplication of fractional analytic functions play important roles in this paper. In fact, our result is a generalization of traditional calculus result.

II. DEFINITIONS AND PROPERTIES

Firstly, we introduce the fractional calculus used in this paper.

Definition 2.1 ([17])**:** Let $0<\alpha\leq 1$, and x_0 be a real number. The Jumarie's modified Riemann-Liouville (R-L) α-fractional derivative is defined by

$$\left({}_{x_0}D_x^{\alpha}\right)[f(x)] = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{x_0}^{x}\frac{f(t)-f(x_0)}{(x-t)^{\alpha}}dt\ , \tag{2}$$

And the Jumarie type of Riemann-Liouville α-fractional integral is defined by

$$\left({}_{x_0}I_x^{\alpha}\right)[f(x)] = \frac{1}{\Gamma(\alpha)}\int_{x_0}^{x}\frac{f(t)}{(x-t)^{1-\alpha}}dt\ , \tag{3}$$

where $\Gamma(\ \ )$ is the gamma function.

In the following, some properties of Jumarie type of fractional derivative are introduced.

Proposition 2.2 ([18])**:** *If* α, β, x_0, C *are real numbers and* $\beta \geq \alpha > 0$, *then*

$$\left({}_{x_0}D_x^{\alpha}\right)\left[(x-x_0)^{\beta}\right] = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(x-x_0)^{\beta-\alpha}, \tag{4}$$

and

$$\left({}_{x_0}D_x^{\alpha}\right)[C] = 0. \tag{5}$$

Next, we introduce the definition of fractional analytic function.

Definition 2.3 ([19])**:** If x, x_0, and a_k are real numbers for all k, $x_0 \in (a,b)$, and $0 < \alpha \leq 1$. If the function $f_\alpha: [a,b] \to R$ can be expressed as an α-fractional power series, i.e., $f_\alpha(x^\alpha) = \sum_{k=0}^{\infty}\frac{a_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha}$ on some open interval containing x_0, then we say that $f_\alpha(x^\alpha)$ is α-fractional analytic at x_0. Furthermore, if $f_\alpha: [a,b] \to R$ is continuous on closed interval $[a,b]$ and it is α-fractional analytic at every point in open interval (a,b), then f_α is called an α-fractional analytic function on $[a,b]$.

In the following, a new multiplication of fractional analytic functions is introduced.

Definition 2.4 ([20])**:** If $0 < \alpha \leq 1$, and x_0 is a real number. Suppose that $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ are α-fractional analytic at $x = x_0$,

$$f_\alpha(x^\alpha) = \sum_{k=0}^{\infty}\frac{a_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha}, \tag{6}$$

$$g_\alpha(x^\alpha) = \sum_{k=0}^{\infty}\frac{b_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha}. \tag{7}$$

Then

$$\begin{aligned} & f_\alpha(x^\alpha) \otimes g_\alpha(x^\alpha) \\ & = \sum_{k=0}^{\infty}\frac{a_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha} \otimes \sum_{k=0}^{\infty}\frac{b_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha} \\ & = \sum_{k=0}^{\infty}\frac{1}{\Gamma(k\alpha+1)}\left(\sum_{m=0}^{k}\binom{k}{m}a_{k-m}b_m\right)(x-x_0)^{k\alpha}. \end{aligned} \tag{8}$$

In other words,

$$\begin{aligned} & f_\alpha(x^\alpha) \otimes g_\alpha(x^\alpha) \\ & = \sum_{k=0}^{\infty}\frac{a_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes k} \otimes \sum_{k=0}^{\infty}\frac{b_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes k} \\ & = \sum_{k=0}^{\infty}\frac{1}{k!}\left(\sum_{m=0}^{k}\binom{k}{m}a_{k-m}b_m\right)\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes k}. \end{aligned} \tag{9}$$

Definition 2.5 ([21])**:** If $0 < \alpha \leq 1$, and $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ are α-fractional analytic at $x = x_0$,

$$f_\alpha(x^\alpha) = \sum_{k=0}^{\infty}\frac{a_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha} = \sum_{k=0}^{\infty}\frac{a_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes k}, \tag{10}$$

$$g_\alpha(x^\alpha) = \sum_{k=0}^{\infty}\frac{b_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha} = \sum_{k=0}^{\infty}\frac{b_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes k}. \tag{11}$$

The compositions of $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ are defined by

$$(f_\alpha \circ g_\alpha)(x^\alpha) = f_\alpha(g_\alpha(x^\alpha)) = \sum_{k=0}^{\infty} \frac{a_k}{k!}(g_\alpha(x^\alpha))^{\otimes k}, \tag{12}$$

and

$$(g_\alpha \circ f_\alpha)(x^\alpha) = g_\alpha(f_\alpha(x^\alpha)) = \sum_{k=0}^{\infty} \frac{b_k}{k!}(f_\alpha(x^\alpha))^{\otimes k}. \tag{13}$$

Definition 2.6 ([22])**:** Let $0 < \alpha \le 1$, and x be a real number. The α-fractional exponential function is defined by

$$E_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{x^{k\alpha}}{\Gamma(k\alpha+1)} = \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes k}. \tag{14}$$

Definition 2.7: Let $0 < \alpha \le 1$, and $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ be two α-fractional analytic functions. Then $(f_\alpha(x^\alpha))^{\otimes n} = f_\alpha(x^\alpha) \otimes \cdots \otimes f_\alpha(x^\alpha)$ is called the n-th power of $f_\alpha(x^\alpha)$. On the other hand, if $f_\alpha(x^\alpha) \otimes g_\alpha(x^\alpha) = 1$, then $g_\alpha(x^\alpha)$ is called the $\otimes$ reciprocal of $f_\alpha(x^\alpha)$, and is denoted by $(f_\alpha(x^\alpha))^{\otimes -1}$.

Definition 2.8 ([23])**:** Let $0 < \alpha \le 1$. If $u_\alpha(x^\alpha)$, $w_\alpha(x^\alpha)$ are two α-fractional analytic functions. Then the α-fractional power exponential function $u_\alpha(x^\alpha)^{\otimes w_\alpha(x^\alpha)}$ is defined by

$$u_\alpha(x^\alpha)^{\otimes w_\alpha(x^\alpha)} = E_\alpha\left(w_\alpha(x^\alpha) \otimes Ln_\alpha(u_\alpha(x^\alpha))\right). \tag{15}$$

Theorem 2.9 (integration by parts for fractional calculus) ([24])**:** *Assume that* $0 < \alpha \le 1$, a, b *are real numbers, and* $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ are α-fractional analytic functions, *then*

$$\left({}_aI_b^\alpha\right)\left[f_\alpha(x^\alpha) \otimes \left({}_aD_x^\alpha\right)[g_\alpha(x^\alpha)]\right] = [f_\alpha(x^\alpha) \otimes g_\alpha(x^\alpha)]_{x=a}^{x=b} - \left({}_aI_b^\alpha\right)\left[g_\alpha(x^\alpha) \otimes \left({}_aD_x^\alpha\right)[f_\alpha(x^\alpha)]\right]. \tag{16}$$

Theorem 2.10 (change of variable for fractional calculus) ([25])**:** *If* $0 < \alpha \le 1$, f_α, u_α *are* α*-fractional analytic functions such that the range of* u_α *contained in the domain of* f_α, *then* $f_\alpha \circ u_\alpha$ *is a* α*-fractional analytic function and*

$$\left({}_aI_b^\alpha\right)\left[(f_\alpha \circ u_\alpha)(x^\alpha) \otimes \left({}_aD_x^\alpha\right)[u_\alpha(x^\alpha)]\right] = \left({}_{u_\alpha(a^\alpha)}I_{u_\alpha(b^\alpha)}^\alpha\right)[f_\alpha(u_\alpha)], \tag{17}$$

Theorem 2.11 (fractional L'Hospital's rule) ([26])**:** *Suppose that* $0 < \alpha \le 1$, c *is a real number, and* $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$, $[g_\alpha(x^\alpha)]^{\otimes -1}$ *are* α*-fractional analytic functions at* $x = c$. *If* $\lim_{x\to c} f_\alpha(x^\alpha) = \lim_{x\to c} g_\alpha(x^\alpha) = 0$, *or* $\lim_{x\to c} f_\alpha(x^\alpha) = \pm\infty$, *and* $\lim_{x\to c} g_\alpha(x^\alpha) = \pm\infty$. *Assume that* $\lim_{x\to c} f_\alpha(x^\alpha) \otimes [g_\alpha(x^\alpha)]^{\otimes -1}$ *and* $\lim_{x\to c}\left({}_cD_x^\alpha\right)[f_\alpha(x^\alpha)] \otimes \left[\left({}_cD_x^\alpha\right)[g_\alpha(x^\alpha)]\right]^{\otimes -1}$ *exist,* $\left({}_cD_x^\alpha\right)[g_\alpha(x^\alpha)](c) \neq 0$. *Then*

$$\lim_{x\to c} f_\alpha(x^\alpha) \otimes [g_\alpha(x^\alpha)]^{\otimes -1} = \lim_{x\to c}\left({}_cD_x^\alpha\right)[f_\alpha(x^\alpha)] \otimes \left[\left({}_cD_x^\alpha\right)[g_\alpha(x^\alpha)]\right]^{\otimes -1}. \tag{18}$$

III. MAIN RESULT

In this section, we obtain the main result in this paper. At first, we need two lemmas.

Lemma 3.1: *Let* $0 < \alpha \le 1$ *and* n *be a non-negative integer. Then the improper* α*-fractional integral*

$$\left({}_0I_{+\infty}^\alpha\right)\left[\left(\frac{1}{\Gamma(\alpha+1)}t^\alpha\right)^{\otimes n} \otimes E_\alpha(-t^\alpha)\right] = n! \tag{19}$$

Proof If $n = 0$, then

$$\left({}_0I_{+\infty}^\alpha\right)[E_\alpha(-t^\alpha)] = -E_\alpha(-t^\alpha)|_0^{+\infty} = 1. \tag{20}$$

Furthermore, by integration by parts for fractional calculus,

$$\left({}_0I_{+\infty}^\alpha\right)\left[\left(\frac{1}{\Gamma(\alpha+1)}t^\alpha\right)^{\otimes n} \otimes E_\alpha(-t^\alpha)\right]$$

$$= \left({}_0I_{+\infty}^\alpha\right)\left[\left(\frac{1}{\Gamma(\alpha+1)}t^\alpha\right)^{\otimes n} \otimes \left({}_0D_t^\alpha\right)[-E_\alpha(-t^\alpha)]\right]$$

$$= -\left(\frac{1}{\Gamma(\alpha+1)}t^{\alpha}\right)^{\otimes n} E_{\alpha}(-t^{\alpha})|_0^{+\infty} + n\cdot\left({}_0I_{+\infty}^{\alpha}\right)\left[\left(\frac{1}{\Gamma(\alpha+1)}t^{\alpha}\right)^{\otimes(n-1)} \otimes E_{\alpha}(-t^{\alpha})\right]$$

$$= n\cdot\left({}_0I_{+\infty}^{\alpha}\right)\left[\left(\frac{1}{\Gamma(\alpha+1)}t^{\alpha}\right)^{\otimes(n-1)} \otimes E_{\alpha}(-t^{\alpha})\right]. \text{ (by fractional L'Hospital's rule)} \quad (21)$$

Therefore, by induction, we obtain the desired result. Q.e.d.

Lemma 3.2: *Let* $0 < \alpha \leq 1$ *and* n *be a non-negative integer. Then the improper* α*-fractional integral*

$$\left({}_0I^{\alpha}_{[\Gamma(\alpha+1)]^{\frac{1}{\alpha}}}\right)\left[\left[-\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)\otimes Ln_{\alpha}(x^{\alpha})\right]^{\otimes n}\right] = \frac{n!}{(n+1)^{n+1}}. \quad (22)$$

Proof Let $\frac{1}{\Gamma(\alpha+1)}y^{\alpha} = -Ln_{\alpha}(x^{\alpha})$, then $\frac{1}{\Gamma(\alpha+1)}x^{\alpha} = E_{\alpha}(-y^{\alpha})$. Thus, by change of variable for fractional calculus,

$$\left({}_0I^{\alpha}_{[\Gamma(\alpha+1)]^{\frac{1}{\alpha}}}\right)\left[\left[-\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)\otimes Ln_{\alpha}(x^{\alpha})\right]^{\otimes n}\right]$$

$$= -\left({}_0I_{+\infty}^{\alpha}\right)\left[\left[\left(E_{\alpha}(-y^{\alpha})\right)\otimes\frac{1}{\Gamma(\alpha+1)}y^{\alpha}\right]^{\otimes n}\otimes -E_{\alpha}(-y^{\alpha})\right]$$

$$= \left({}_0I_{+\infty}^{\alpha}\right)\left[\left(E_{\alpha}(-y^{\alpha})\right)^{\otimes n}\otimes\left(\frac{1}{\Gamma(\alpha+1)}y^{\alpha}\right)^{\otimes n}\otimes E_{\alpha}(-y^{\alpha})\right]$$

$$= \left({}_0I_{+\infty}^{\alpha}\right)\left[\left(\frac{1}{\Gamma(\alpha+1)}y^{\alpha}\right)^{\otimes n}\otimes E_{\alpha}(-(n+1)y^{\alpha})\right]$$

$$= \frac{1}{(n+1)^{n+1}}\left({}_0I_{+\infty}^{\alpha}\right)\left[\left(\frac{1}{\Gamma(\alpha+1)}t^{\alpha}\right)^{\otimes n}\otimes E_{\alpha}(-t^{\alpha})\right] \quad \left(\text{let } \frac{1}{\Gamma(\alpha+1)}t^{\alpha} = (n+1)\frac{1}{\Gamma(\alpha+1)}y^{\alpha}\right)$$

$$= \frac{n!}{(n+1)^{n+1}}. \quad \text{(by Lemma 3.1)}$$

Q.e.d.

Theorem 3.3: *If* $0 < \alpha \leq 1$, *then the improper* α*-fractional integral*

$$\left({}_0I^{\alpha}_{[\Gamma(\alpha+1)]^{\frac{1}{\alpha}}}\right)\left[\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes -\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)}\right] = \sum_{n=1}^{\infty}\frac{1}{n^n}. \quad (23)$$

Proof

$$\left({}_0I^{\alpha}_{[\Gamma(\alpha+1)]^{\frac{1}{\alpha}}}\right)\left[\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes -\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)}\right]$$

$$= \left({}_0I^{\alpha}_{[\Gamma(\alpha+1)]^{\frac{1}{\alpha}}}\right)\left[E_{\alpha}\left(-\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)\otimes Ln_{\alpha}(x^{\alpha})\right)\right]$$

$$= \left({}_0I^{\alpha}_{[\Gamma(\alpha+1)]^{\frac{1}{\alpha}}}\right)\left[\sum_{n=0}^{\infty}\frac{1}{n!}\left[-\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)\otimes Ln_{\alpha}(x^{\alpha})\right]^{\otimes n}\right]$$

$$= \sum_{n=0}^{\infty}\frac{1}{n!}\left({}_0I^{\alpha}_{[\Gamma(\alpha+1)]^{\frac{1}{\alpha}}}\right)\left[\left[-\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)\otimes Ln_{\alpha}(x^{\alpha})\right]^{\otimes n}\right]$$

$$= \sum_{n=0}^{\infty}\frac{1}{n!}\cdot\frac{n!}{(n+1)^{n+1}} \quad \text{(by Lemma 3.2)}$$

$$= \sum_{n=0}^{\infty}\frac{1}{(n+1)^{n+1}}$$

$$= \sum_{n=1}^{\infty}\frac{1}{n^n}.$$

Q.e.d.

IV. CONCLUSION

In this paper, based on Jumarie type of R-L fractional calculus, we solve some type of improper fractional integral mainly using change of variable for fractional calculus, integration by parts for fractional calculus, and fractional L'Hospital's rule. A new multiplication of fractional analytic functions plays an important role in this paper. In fact, our result is a generalization of the result in ordinary calculus. In the future, we will continue to use these methods to study the problems in applied mathematics and fractional differential equations.

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